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Numerical Solution of the Eulerian Equations of Compressible Flow by a Finite Element Method Which Follows the Free Boundary and the Interfaces

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We describe a numerical method to compute the solution of a system of conservation laws subject to an initial condition and certain boundary conditions which involve a free boundary. This method is specially designed to solve the Eulerian equations of compressible flow. It is based on finite elements relative to the space variables *and* the time; the elements near the boundary (or the interfaces) are chosen in such a way that they follow the boundary (or the interfaces); the choice of the interior elements is arbitrary at each time step. For the computation of discontinuous solutions (shocks), we introduce a special type of pseudoviscosity. Numerical results are given.

I. INTRODUCTION

The numerical solution of the Eulerian equations of hydrodynamics by finite difference methods exhibits serious difficulties for the treatment of free boundaries and interfaces, since these surfaces generally do not go through the mesh points. In this paper, we describe a finite element method which avoids this inconvenience; it is a variant of a method used previously by the authors for the Stefan problem [2]; for greater simplicity, we describe it in the case of one space variable, but the extension to the multidimensional case is straightforward.

The elements are polyhedrals of the space–time space; those near the boundary (or interfaces) are chosen in such a way that some of their vertices are located *on* the boundary (or interfaces); the choice of the interior elements is arbitrary at each time step, which leads to a great flexibility in the application of the method.

The method that we describe in this paper is the most elementary method based on space–time finite elements; many variants can be derived by using various types of interpolating functions.

Let us also mention that our method presents a vague analogy with the method used by Amsden and Hirt in the code YAQUI [1].

In Section II, we introduce the problems that we want to solve; we consider a general system of conservation laws and the particular cases of the Eulerian equations of hydrodynamics for an isentropic flow and for a nonisentropic flow. We transform the system of partial differential equations into an integral identity which will be the basis of our method.

In Section III, we describe our method in the simple case of a single conservation law. In the particular case of rectangular elements, our method turns out to be identical to the well-known Crank–Nicolson scheme.

In Section IV, we describe our method for a general system of conservation laws and we examine in detail the application to hydrodynamics.

In Section V, we introduce a special pseudoviscosity term which is necessary for the computation of discontinuous solutions.

Finally, Section VI is devoted to numerical experiments for three different problems: Burgers equation (scalar case), isentropic flow (system of two equations), and nonisentropic flow (system of three equations).

II. Conservation Laws and Hydrodynamics Equations

IIa. *Generalities*

Let us consider a system of conservation laws of the form:

$$(\partial V/\partial t) + (\partial/\partial x)\, F(V) = 0, \tag{2.1}$$

where x is the space variable (1-dimensional case), t the time variable, V an unknown vector-valued function of (x, t), and $F(V)$ a given vector-valued differentiable function of V. We denote by $v^{(s)}$ and $f^{(s)}$, $1 \leqslant s \leqslant m$, the components of V and $F(V)$.

We want to solve (2.1) in a region $\mathscr{R}$ of the form

$$\mathscr{R} = \{(x, t);\, 0 < x < a(t),\, t > 0\},$$

where $a(t)$ is a certain function of t (see Fig. 1). The function $a(t)$ can either be given (*fixed* boundary) or depend on the solution V (*free* boundary).

We impose on the function V an initial condition

$$V(x, 0) = V^0(x), \tag{2.2}$$

where $V^0(x)$ is a given function of x, and boundary conditions which depend on the function $F(V)$ in the following way. Let $F'(V)$ be the matrix with coefficients $\alpha_{sr} = \partial f^{(s)}/\partial v^{(r)}$ and let $\lambda_s(V)$ be its eigenvalues. The curves $\mathscr{C}_s$ which satisfy

$$dx/dt = \lambda_s(V),$$

are the characteristic curves of (2.1); let us orient them towards increasing values of t. Let P be a point of the boundary, either the left part of the boundary ($x = 0$) or the right part of the boundary ($x = a(t)$), and let m' be the number of characteristic curves through P which enter the region $\mathscr{R}$, $0 \leqslant m' \leqslant m$. Then, at the point P we impose m' scalar boundary conditions on the function V (see Fig. 1).

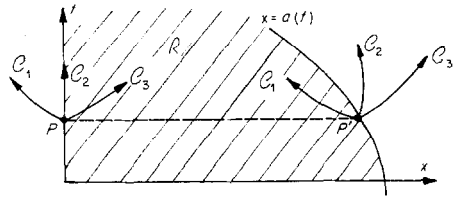

FIG. 1. Characteristic curves and boundary conditions. In this case we have: $m = 3$, $m'(P) = 1$, $m'(P') = 1$; we must impose one boundary condition at P and one at P'.

We will now write the system (2.1) in the form of an *integral identity*. Let G be the part of $\mathscr{R}$ which lies between the times $t = \tau_1$ and $t = \tau_2$, $0 \leqslant \tau_1 < \tau_2$, and let ∂G be the boundary of G counter-clockwise oriented. Let $\Phi(x, t)$ be a vector-valued function defined and continuous in $\bar{G}$ and which admits bounded (possibly discontinuous) first derivatives. Taking the inner product of (2.1) with Φ and integrating by parts in G. we get

$$\iint_G V \cdot \frac{\partial \Phi}{\partial t}\, dx\, dt + \iint_G F(V) \cdot \frac{\partial \Phi}{\partial x}\, dx\, dt + \int_{\partial G} \Phi \cdot (V\, dx - F(V)\, dt) = 0, \quad \forall \Phi. \tag{2.3}$$

This identity can be written in a slightly different form if we denote by C_0, C_1, C_2, C_a the parts of ∂G corresponding to $x = 0$, $t = \tau_1$, $t = \tau_2$, $x = a(t)$, respectively, and oriented towards increasing x or t (see Fig. 2). Then, (2.3) becomes

$$\begin{aligned} &\iint_G V \cdot \frac{\partial \Phi}{\partial t}\, dx\, dt + \iint_G F(V) \cdot \frac{\partial \Phi}{\partial x}\, dx\, dt \\ &\quad - \int_{C_2} \Phi \cdot V\, dx + \int_{C_1} \Phi \cdot V\, dx + \int_{C_0} \Phi \cdot F(V)\, dt \\ &\quad + \int_{C_a} \Phi \cdot (V\, dx - F(V)\, dt) = 0, \quad \forall \Phi. \end{aligned} \tag{2.4}$$

By choosing Φ identical to arbitrary constant vectors, we get

$$\int_{C_2} V\, dx - \int_{C_1} V\, dx = \int_{C_a} (V\, dx - F(V)\, dt) - \int_{C_0} (V\, dx - F(V)\, dt) \tag{2.5}$$

(relation of conservation).

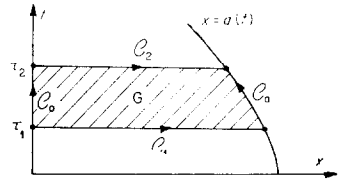

FIG. 2. The domain of integration G and the oriented portions of its boundary.

Finally, let us remark that the integral identity (2.3) makes sense even for functions V which are not differentiable nor continuous. Any function V which satisfies (2.3) is called a generalized or *weak solution* of (2.1). It is easy to prove the following classical result. If a weak solution V is discontinuous along a line $\mathscr{C}$, then we have, along $\mathscr{C}$,

$$[V]\,dx - [F(V)]\,dt = 0, \tag{2.6}$$

where $[V]$ and $[F(V)]$ denote the jumps of the functions V and $F(V)$ across the line $\mathscr{C}$ (see [5]).

IIb. *Eulerian Equations of Compressible Flow*

We consider a compressible nonviscous fluid lying in the slab $0 < x < a(t)$. We assume that body forces, heat conduction, and energy sources are absent. We use the following notations: ρ is the density, u is the speed (in the direction of the space-variable x), E is the total energy per unit-volume and p is the pressure. The pressure is related to ρ, u and E by the equation of state of the fluid:

$$p = p(\rho, \epsilon), \tag{2.7}$$

where $\epsilon = E - \frac{1}{2}\rho u^2$ and $p(\rho, \epsilon)$ is a given function of ρ and ϵ.

The Eulerian equations of the flow, for slab symmetry, are expressed in the form (2.1), where

$$V = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \quad \text{and} \quad F(V) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ Eu + pu \end{pmatrix}. \tag{2.8}$$

We can also write $F(V) = uV + H(V)$, where

$$H(V) = \begin{pmatrix} 0 \\ p \\ pu \end{pmatrix}.$$

The function $x = a(t)$ is not given; it will be determined by the condition that the corresponding free boundary follows the movement of the fluid, which yields

$$a'(t) = u(a(t), t). \tag{2.9}$$

On the free boundary, we impose a condition on the pressure

$$p(a(t), t) = \pi(t), \tag{2.10}$$

where $\pi(t)$ is a given positive function of t.

On the fixed boundary $x = 0$, we impose a condition on the speed

$$u(0, t) = 0. \tag{2.11}$$

It is easy to verify that the boundary conditions (2.10) and (2.11) are in accordance with the criterion given in IIa; the slopes of the characteristic curves are u, $u \pm c$, where $c = ((\partial p/\partial \rho) + [(p + \epsilon)/\rho](\partial p/\partial \epsilon))^{1/2} > 0$ is the sound speed; it follows that we need one boundary condition on the left (for $x = 0$) and one on the right (for $x = a(t)$).

Since $dx - u\,dt = 0$ on C_0 and C_a, the integral identity (2.4) can be written as

$$\iint_G V \cdot \frac{\partial \Phi}{dt}\,dx\,dt + \iint_G F(V) \cdot \frac{\partial \Phi}{\partial x}\,dx\,dt$$
$$- \int_{C_2} \Phi \cdot V\,dx + \int_{C_1} \Phi \cdot V\,dx + \int_{C_0} \Phi \cdot H(V)\,dt - \int_{C_a} \Phi \cdot H(V)\,dt = 0, \tag{2.12}$$

and the relation of conservation (2.5) as

$$\int_{C_2} V\,dx - \int_{C_1} V\,dx = -\int_{C_a} H(V)\,dt + \int_{C_0} H(V)\,dt, \tag{2.13}$$

which is a condensed vector-form for the well-known relations of conservation of mass, momentum, and energy.

Finally, the jump condition (2.6) yields the classical Rankine–Hugoniot relations

$$dx/dt = [\rho u]/[u] = [\rho u^2]/[\rho u] = [Eu + p]/[E].$$

Particular Case: *Isentropic Flow*

In this case, ϵ is a function of ρ and therefore the equation of state (2.7) implies that p is a function of ρ alone:

$$p = p(\rho). \tag{2.14}$$

Then, the first two equations are not coupled with the third one; we can write them in the form (2.1) with

$$V = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}, \qquad F(V) = uV + H(V), \qquad H(V) = \begin{pmatrix} 0 \\ p \end{pmatrix}. \tag{2.15}$$

The slopes of the characteristics are $u \pm c$ where $c = (p'(\rho))^{1/2}$. We impose the same boundary conditions as before.

III. Numerical Method for a Single Conservation Law

To facilitate the understanding of our method, we describe it first in the simple case of a single conservation law:

$$(\partial v/\partial t) + (\partial/\partial x) f(v) = 0, \tag{3.1}$$

where $f(v)$ is a given function of v. We want to solve this equation in the region $\mathscr{R}$ defined in the previous section; to simplify further, we assume at first, that the function $x = a(t)$ is given. We impose the initial condition:

$$v(x, 0) = v^0(x) = \text{given function of } x. \tag{3.2}$$

For the boundary conditions, let us assume for example that $f'(v) > 0$ for $x = 0$ and $f'(v) > a'(t)$ for $x = a(t)$. Then, we impose a boundary condition on the left:

$$v(0, t) = g(t) = \text{given function of } t, \tag{3.3}$$

and no boundary condition on the right. The integral form of (3.1) is

$$A_G(v, \varphi) = 0, \qquad \forall \varphi, \tag{3.4}$$

where

$$A_G(v, \varphi) = \iint_G v \frac{\partial \varphi}{\partial t}\, dx\, dt + \iint_G f(v) \frac{\partial \varphi}{\partial x}\, dx\, dt - \int_{C_2} \varphi v\, dx$$
$$+ \int_{C_1} \varphi v\, dx + \int_{C_0} \varphi f(v)\, dt + \int_{C_a} \varphi(v\, dx - f(v)\, dt), \tag{3.5}$$

with the same notations as in (2.4).

Taking $\varphi \equiv 1$, we get the relation of conservation:

$$\int_{C_2} v\, dx - \int_{C_1} v\, dx = \int_{C_a} (v\, dx - f(v)\, dt + \int_{C_0} f(v)\, dt. \tag{3.6}$$

To appxoximate the problem (3.1), (3.2), (3.3), we proceed as follows (see [2]). Let I be a positive integer. We consider a set of points $P_i^n = (x_i^n, t^n)$ for $0 \leqslant i \leqslant I$ and $n \geqslant 0$, where

$$0 = t^0 \leqslant t^n < t^{n+1} \qquad \text{and} \qquad 0 = x_0^n \leqslant x_i^n < x_{i+1}^n \leqslant x_I^n = a(t^n)$$

for $0 \leqslant i \leqslant I-1$ and $n \geqslant 0$. We denote by K_i^n the trapezoid with vertices P_i^n, P_{i+1}^n, P_{i+1}^{n+1}, P_i^{n+1} (see Fig. 3). Let h be an unspecified parameter wich characterizes the grid formed by the points P_i^n and which converges to zero as the grid is refined. Let $\mathscr{R}_h = \bigcup_{n\geqslant 0} \bigcup_{i=0}^{I-1} K_i^n$. We denote by $\mathscr{V}_h$ the space of all continuous functions defined on $\mathscr{R}_h$ which are linear along the four sides of each trapezoid K_i^n and which are linear with respect to x inside each trapezoid K_i^n; the functions of $\mathscr{V}_h$ are uniquely determined by their values at the points P_i^n. Let $\mathscr{E}_h$ be the subspace of all $\varphi \in \mathscr{V}_h$ such that $\varphi(P_i^n)$ is independent of n; $\mathscr{E}_h$ is a space of dimension $I+1$.

Our numerical method is based on a discrete analog of (3.4) with $G = G^n =$ the region G obtained for $\tau_1 = t^n$ and $\tau_2 = t^{n+1}$. Let $A^n(v, \varphi)$ be an approximation of $A_{G^n}(v, \varphi)$ obtained by replacing G^n by $\tilde{G}^n = \bigcup_{i=0}^{I-1} K_i^n$ and by computing the various integrals involved by means of numerical quadrature formulae that we will specify later. Then, we approximate the problem (3.1), (3.2), (3.3) by the following.

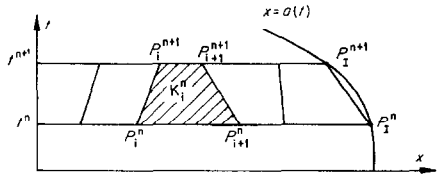

FIG. 3. The discretization of the region $\mathscr{R}$.

Discrete problem:

Find $v_h \in \mathscr{V}_h$ such that:

$$v_h(P_i^0) = v^0(x_i^0), \qquad \text{for} \quad 0 \leqslant i \leqslant I, \tag{3.7}$$

$$v_h(P_0^n) = g(t^n), \qquad \text{for} \quad n > 0, \tag{3.8}$$

$$A^n(v_h, \varphi) = 0, \qquad \forall \varphi \in \mathscr{S}_h, \qquad \forall n \geqslant 0, \tag{3.9}$$

where $\mathscr{S}_h$ is a certain subspace of $\mathscr{E}_h$ which depends on the boundary conditions of the problem and that we are going to specify.

For each n the relation (3.9) represents a system of nonlinear algebraic equations that we can write in terms of the values of the unknown function v_h at the grid points $\{P_i^n\}_{i=0}^I$ and $\{P_i^{n+1}\}_{i=0}^I$; the number of these equations is equal to the dimension of the subspace $\mathscr{S}_h$. Suppose that we have computed v_h until the time t^n; we want (3.9) to permit the computation of v_h at the time t^{n+1}; taking into account the boundary condition (3.8), there are I unknowns: the values of v_h at the points P_i^{n+1} for $1 \leqslant i \leqslant I$; so, we need I equations and, consequently, *$\mathscr{S}_h$ must be of dimension I.*

Moreover, we want v_h to satisfy the discrete analog of the conservation relation (3.6), obtained by letting $\varphi \equiv 1$ in (3.9) in the same way as (3.6) was obtained by letting $\varphi \equiv 1$ in (3.4). For this reason, we will impose to $\mathscr{S}_h$ the following condition: *$\mathscr{S}_h$ must contain the function φ which is identical to* 1.

For each $i \in \{0, 1, \ldots, I\}$, let $\varphi^{(i)}$ be the function of $\mathscr{E}_h$ such that:

$$\varphi^{(i)}(P_j^n) = \delta_{ij} = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases} \tag{3.10}$$

The functions $\varphi^{(i)}$ form a basis of $\mathscr{E}_h$ and they satisfy

$$\sum_{i=0}^{I} \varphi^{(i)} \equiv 1. \tag{3.11}$$

We will choose for $\mathscr{S}_h$ the subspace $\mathscr{E}_h^L$ generated by the set of functions $\{\varphi^{(0)} + \varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}, \ldots, \varphi^{(I)}\}$. This space satisfies the two conditions stated above.

If we write (3.9) for each of the basis functions of $\mathscr{S}_h$, the resulting system of algebraic equations has a *tridiagonal* structure for the unknowns $v_1^{n+1}, v_2^{n+1}, \ldots, v_I^{n+1}$ (note that the known value v_0^{n+1} appears in one equation, while the unknown value v_I^{n+1} on the right boundary appears in two equations).

Remark 3.1. In the case when we must impose a boundary condition on the right, instead of on the left, we choose for $\mathscr{S}_h$ the subspace $\mathscr{E}_h^R$ generated by $\{\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(I-2)}, \varphi^{(I-1)} + \varphi^{(I)}\}$. In the case when we must impose a boundary condition on both sides, i.e., on the left and on the right, we choose for $\mathscr{S}_h$ the sub-

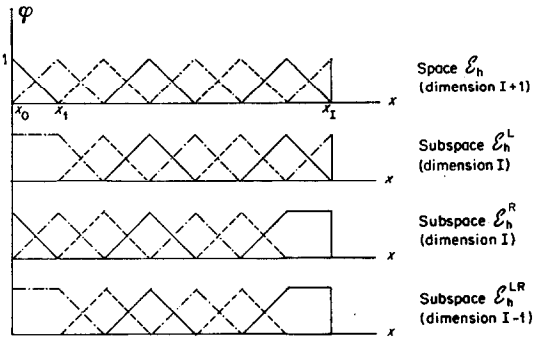

FIG. 4. The basis functions of the spaces $\mathscr{E}_h$, $\mathscr{E}_h^L$, $\mathscr{E}_h^R$ and $\mathscr{E}_h^{LR}$. The choice of $\mathscr{S}_h$ depends on the boundary conditions as follows.

Case 1: *no* boundary condition: $\mathscr{S}_h = \mathscr{E}_h$.
Case 2: one boundary condition on the *left*: $\mathscr{S}_h = \mathscr{E}_h^L$.
Case 3: one boundary condition on the *right*: $\mathscr{S}_h = \mathscr{E}_h^R$.
Case 4: one boundary condition on *each* side: $\mathscr{S}_h = \mathscr{E}_h^{LR}$.

space $\mathscr{E}_h^{LR}$ generated by $\{\varphi^{(0)} + \varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(I-2)}, \varphi^{(I-1)} + \varphi^{(I)}\}$. Finally, when we must impose no boundary condition at all, we choose $\mathscr{S}_h = \mathscr{E}_h$. The basis functions of the spaces $\mathscr{E}_h$, $\mathscr{E}_h^L$, $\mathscr{E}_h^R$ and $\mathscr{E}_h^{LR}$ are represented on Fig. 4.

Choice of the Numerical Quadrature Formulae

To complete the description of our method, it remains to specify the numerical quadrature formulae that we use to approximate the integrals appearing in the expression $A_{G^n}(v, \varphi)$. The line integrals are computed by using the *trapezoidal rule* on each of the elementary segments joining two neighbor grid points. The surface integrals are computed by using the following formula on each of the trapezoids K_i^n. For any continuous function ψ, the integral

$$I_{K_i^n}(\psi) = \iint_{K_i^n} \psi(x, t)\, dx\, dt$$

is approximated by

$$\tilde{I}_{K_i^n}(\psi) = \tfrac{1}{4}(t^{n+1} - t^n)[(x_{i+1}^n - x_i^n)(\psi_i^n + \psi_{i+1}^n) + (x_{i+1}^{n+1} - x_i^{n+1})(\psi_i^{n+1} + \psi_{i+1}^{n+1})]. \tag{3.12}$$

Note that $\tilde{I}_{K_i^n}(\psi) = I_{K_i^n}(\psi)$ for all $\psi \in \mathscr{V}_h$.

Explicit Expression of the Discrete Equations

After easy computations such as those which have been developed in [2], we get the following expressions for Eqs. (3.9), with the notations $v_i^n = v_h(P_i^n)$ and $f_i^n = f(v_i^n)$.

For $1 < i < I$,

$$\begin{aligned} &-\tfrac{1}{4}[(x_{i+1}^{n+1} - x_{i+1}^n)(v_{i+1}^n + v_{i+1}^{n+1}) - (x_{i-1}^{n+1} - x_{i-1}^n)(v_{i-1}^n + v_{i-1}^{n+1})] \\ &\quad + \tfrac{1}{2}[(x_{i+1}^{n+1} - x_{i-1}^{n+1})\, v_i^{n+1} - (x_{i+1}^n - x_{i-1}^n)\, v_i^n] \\ &\quad + \tfrac{1}{4}(t^{n+1} - t^n)(f_{i+1}^n + f_{i+1}^{n+1} - f_{i-1}^n - f_{i-1}^{n+1}) = 0. \end{aligned} \tag{3.13}$$

For $i = 1$,

$$\begin{aligned} &-\tfrac{1}{4}[(x_{i+1}^{n+1} - x_{i+1}^n)(v_{i+1}^n + v_{i+1}^{n+1}) + (x_i^{n+1} - x_i^n)(v_i^n + v_i^{n+1})] \\ &\quad + \tfrac{1}{2}[(x_{i+1}^{n+1} - x_{i-1}^{n+1})\, v_i^{n+1} - (x_{i+1}^n - x_{i-1}^n)\, v_i^n] \\ &\quad + \tfrac{1}{2}[(x_i^{n+1} - x_{i-1}^{n+1})\, v_{i-1}^{n+1} - (x_i^n - x_{i-1}^n)\, v_{i-1}^n] \\ &\quad + \tfrac{1}{4}(t^{n+1} - t^n)[(f_i^n + f_i^{n+1} + f_{i+1}^n + f_{i+1}^{n+1}) - 2(f_{i-1}^n + f_{i-1}^{n+1})] = 0. \end{aligned} \tag{3.14}$$

For $i = I$,

$$\tfrac{1}{4}[(x_{i-1}^{n+1} - x_{i-1}^{n})(v_{i-1}^{n} + v_{i-1}^{n+1}) - (x_i^{n+1} - x_i{}^{n})(v_i{}^{n} + v_i^{n+1})$$
$$+ \tfrac{1}{2}[(x_i^{n+1} - x_{i-1}^{n+1})\, v_i^{n+1} - (x_i{}^{n} - x_{i-1}^{n})\, v_i{}^{n}]$$
$$+ \tfrac{1}{4}(t^{n+1} - t^{n})(f_i{}^{n} + f_i^{n+1} - f_{i-1}^{n} - f_{i-1}^{n+1}) = 0. \tag{3.15}$$

Remark 3.2. In the particular case of equal rectangular elements, i.e., when $x_i{}^{n} = x_i^{n+1} = x_{i-1}^{n} + h = x_{i-1}^{n+1} + h = x_{i+1}^{n} - h = x_{i+1}^{n+1} - h$, Eq. (3.13) becomes, after setting $t^{n+1} - t^{n} = k$ and dividing by $-hk$,

$$[(v_i^{n+1} - v_i{}^{n})/k] + \tfrac{1}{2}\,([(f_{i+1}^{n} - f_{i-1}^{n})/2h] + [(f_{i+1}^{n+1} - f_{i-1}^{n+1}/2h]) = 0, \tag{3.16}$$

which is the classical Crank–Nicolson scheme.

Remark 3.3. In the case of rectangular elements, our method is also the same as the method of Swartz and Wendroff [10] except for the use of numerical quadrature formulae; we get their method if we compute exactly the integrals relative to x and use the trapezoidal rule to approximate the integrals relative to t.

Numerical solution of the System of Algebraic Equations

The system of algebraic equations (3.9) is of the form

$$M^{n}\mathbf{v}^{n+1} + (t^{n+1} - t^{n})\, N\mathbf{f}^{n+1} = \mathbf{d}^{n}, \tag{3.17}$$

where $\mathbf{v}^{n+1}$ denotes the vector of components $\{v_i^{n+1}\}_{i=1}^{I}$, $\mathbf{f}^{n+1}$ denotes the vector of components $\{f_i^{n+1}\}_{i=1}^{I}$, M^{n} is a tridiagonal matrix which depends only on the trapezoids $K_i{}^{n}$, N is a constant tridiagonal matrix (with coefficients $0, \pm\frac{1}{4}$) and $\mathbf{d}^{n}$ is a vector which depends only on known quantities at the time t^{n} and on the given boundary value v_0^{n+1}.

Given all the values at time t^{n}, suppose that we choose the trapezoids $K_i{}^{n}$ so that

$$| x_i^{n+1} - x_i{}^{n} | < c(t^{n+1} - t^{n}), \quad \forall i, \tag{3.18}$$

where c is a certain constant. Then, the matrix M^{n} is diagonally dominant for $(t^{n+1} - t^{n})$ small enough and converges to a diagonal matrix as $(t^{n+1} - t^{n})$ converges to zero.

To solve the system (3.17), we perform the following iterations.

$$M^{n}\mathbf{v}^{n+1,l+1} + (t^{n+1} - t^{n})\, N\mathbf{f}^{n+1,l} = \mathbf{d}^{n}, \tag{3.19}$$

where l denotes the iteration index. To start the iterations we take $\mathbf{v}^{n+1,0} = \mathbf{v}^{n}$. The method converges for $(t^{n+1} - t^{n})$ small enough (relative to the spatial mesh and to the derivative of f).

Treatment of a Free Boundary

Let us now assume that the function $a(t)$ is not given but depends on the solution v by a relation of the form

$$a'(t) = b(v(a(t), t)), \tag{3.20}$$

where $b(v)$ is a given function of v.

Let $a^n = a_h(t^n)$ denote the approximation of $a(t^n)$. At each time step, we must compute a^{n+1} and $\mathbf{v}^{n+1}$. The trapezoids K_i^n and, therefore, the matrix M^n depend on a^{n+1}. We approximate (3.20) by

$$a^{n+1} = a^n + \tfrac{1}{2}(t^{n+1} - t^n)(b_I^n + b_I^{n+1}), \tag{3.21}$$

where $b_I^n = b(v_I^n)$, and we perform the iterations

$$\begin{aligned} a^{n+1,l+1} &= a^n + \tfrac{1}{2}(t^{n+1} - t^n)(b_I^n + b_I^{n+1,l}), \\ M^{n,l+1}\mathbf{v}^{n+1,l+1} &+ (t^{n+1} - t^n)\, N\mathbf{f}^{n+1,l} = \mathbf{d}^n \end{aligned} \tag{3.22}$$

with $\mathbf{v}^{n+1,0} = \mathbf{v}^n$.

IV. Numerical Method for a System of Conservation Laws

IVa. *General Case*

Let us now consider the general system (2.1) with the initial condition (2.2) and appropriate boundary conditions which depend on $F'(V)$, say m_1 scalar conditions on the left boundary ($x = 0$) and m_2 scalar conditions on the right boundary ($x = a(t)$), with $0 \leqslant m_1 \leqslant m$ and $0 \leqslant m_2 \leqslant m$. We assume at first that the function $a(t)$ is given.

As in the scalar case of Section III, our method is based on a discrete analog of the integral identity (2.4). Let $A^n(V, \Phi)$ be an approximation of the left-hand side member of (2.4) with $G = G^n$. We approximate our problem by the following.

Discrete problem.

Find a function $V_h \in (\mathscr{V}_h)^m$ which satisfies the initial condition at the points P_i^0, the boundary conditions at the points P_0^n and P_I^n, and the condition

$$A^n(V_h, \Phi) = 0, \qquad \forall \Phi \in \mathscr{Y}_h, \qquad \forall n \geqslant 0, \tag{4.1}$$

where $\mathscr{Y}_h$ is a subspace of $(\mathscr{E}_h)^m$ which depends on the boundary conditions of the problem that we are going to specify.

The condition (4.1) must permit the computation of V_h at the time t^{n+1} once

we know the values of V_h at the time t^n. Taking into account the boundary conditions, we need $(m(I+1) - m_1 - m_2)$ equations; hence, the space $\mathscr{Y}_h$ must be of dimension $(m(I+1) - m_1 - m_2)$. Moreover, in order that the approximate solution V_h satisfy the discrete analog of the relation of conservation (2.5), we require $\mathscr{Y}_h$ to contain all constant m-vectors. These considerations lead to the following choice of the space $\mathscr{Y}_h$.

Let us write $\mathscr{Y}_h = \mathscr{Y}_h^{(1)} \times \mathscr{Y}_h^{(2)} \times \cdots \times \mathscr{Y}_h^{(m)}$. We will choose the spaces $\mathscr{Y}_h^{(s)}$ so that

(i) $\mathscr{Y}_h^{(s)} \in \{\mathscr{E}_h, \mathscr{E}_h{}^L, \mathscr{E}_h{}^R, \mathscr{E}_h^{LR}\}$, for each s.

(ii) The total number of superscripts L (respectively R) which appear in all the spaces $\mathscr{Y}_h^{(s)}$ must be equal to m_1 (respectively m_2).

(iii) If the left boundary value of a certain component $v^{(s)}$ is imposed by one of the boundary conditions, then the superscript L must appear in $\mathscr{Y}_h^{(s)}$ for the same value of s, i.e., we must take $\mathscr{Y}_h^{(s)} \in \{\mathscr{E}_h{}^L, \mathscr{E}_h^{LR}\}$. A similar condition is imposed on the right.

The foregoing rules do not in general determine $\mathscr{Y}_h$ uniquely. In general there will remain several possible choices which correspond to different treatments of the boundary conditions (See IVc).

IVb. *Application to Isentropic Flow*

We consider the particular case (2.15) with the pressure p given by (2.14). The boundary conditions are (2.10) and (2.11) together with (2.9). On the right, the value of $p = p(\rho)$ is imposed by (2.10) and therefore the value of $v^{(1)} = \rho$ is imposed (we assume that the function $p(\rho)$ is strictly increasing); on the left, the

(i), (ii), and (iii) of the previous paragraph for the choice of $\mathscr{Y}_h$, it follows that we must take

$$\mathscr{Y}_h = \mathscr{E}_h{}^R \times \mathscr{E}_h{}^L. \tag{4.2}$$

In this case, the choice of $\mathscr{Y}_h$ is unique.

For the computation of the expression $A^n(V, \Phi)$ we use the form (2.12) of the integral identity (2.4) in which we have taken account of the exact relation

$$\int_{C_a} \Phi \cdot V(dx - u\,dt) = 0.$$

For the computation of the free boundary, we approximate (2.9) by

$$a^{n+1} = a^n + \tfrac{1}{2}(t^{n+1} - t^n)(u_I{}^n + u_I^{n+1}). \tag{4.3}$$

For the numerical solution of the resulting system of algebraic equations, we use an iterative method which is similar to the method (3.22) described in the previous section. At each iteration, we must solve a tridiagonal system of linear equations.

IVc. *Application to Nonisentropic Flow*

Now we take V and $F(V)$ as defined by (2.8) with component p given by (2.7). The boundary conditions are the same as before; the value of the $v^{(2)} = \rho u$ on the left boundary is imposed by (2.11); however, condition (2.10) involves the three components of the vector V since the pressure p is now given by (2.7) (instead of (2.14) in IVb).

According to the criterions (i), (ii), and (iii) of IVa, the superscript L must appear in $\mathscr{Y}_h^{(2)}$ and the superscript R must appear in one of the spaces $\mathscr{Y}_h^{(s)}$. This leads to the three following possible choices of $\mathscr{Y}_h$.

$$\text{Either:} \quad \mathscr{Y}_h = \mathscr{E}_h \times \mathscr{E}_h{}^L \times \mathscr{E}_h{}^R, \tag{4.4a}$$

$$\text{Or:} \quad \mathscr{Y}_h = \mathscr{E}_h{}^R \times \mathscr{E}_h{}^L \times \mathscr{E}_h, \tag{4.4b}$$

$$\text{Or:} \quad \mathscr{Y}_h = \mathscr{E}_h \times \mathscr{E}^{LR} \times \mathscr{E}_h. \tag{4.4c}$$

Each of these choices corresponds to a different treatment of the condition on the free boundary. In the numerical experiments described in Section VI, we have used the space $\mathscr{Y}_h$ defined by (4.4a); the other two possibilities have not been numerically tested.

Treatment of the Interfaces

An interface between two fluids can be treated like a free boundary. The position of the interface at the time t^{n+1} is computed by a formula analogous to (4.3). The space $\mathscr{Y}_h$ of test functions Φ used for each fluid on each side of the interface is the same as the space $\mathscr{Y}_h$ used previously for a single fluid.

V. Introduction of Dissipation

It is well-known that the Crank–Nicolson scheme (3.16) gives rise to oscillations when we use it for the computation of solutions which are discontinuous or rapidly changing. Our method, which is an extension of the Crank–Nicolson scheme, has the same disadvantage. To remedy such a disadvantage it is classical to introduce a dissipative effect (pseudoviscosity) in the numerical scheme (see [9]); the choice of a suitable pseudoviscosity depends on the numerical scheme that we are using. We will describe here a procedure that we have used to introduce dissipation in our scheme; it consists in the application of a (possibly iterated) nonlinear

smoothing formula, which preserves the l_1 norm, at each time step. We consider first the case of a single conservation law like in Section III.

At each time step, we add a corrective term $Q\mathbf{v}^{n+1}$ to the vector $\mathbf{v}^{n+1}$ computed by the method of Section III, i.e., the vector $\mathbf{v}^{n+1}$ is replaced by the vector

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}^{n+1} + Q\mathbf{v}^{n+1}, \tag{5.1}$$

where Q is a nonlinear operator which is defined as follows. Let $w(x)$ be an arbitrary function of x and let $\mathbf{w}$ be the vector with components $w_i = w(x_i)$, $i = 0, 1, \ldots, I$. Let $\hat{w}_i$ be the value at x_i of the linear function which interpolates w between the points x_{i-1} and x_{i+1}, i.e.,

$$\hat{w}_i = ((x_{i+1} - x_i)\, w_{i-1} + (x_i - x_{i-1})\, w_{i+1})/(x_{i+1} - x_{i-1}) \tag{5.2}$$

and let

$$\delta^2 w_i = \hat{w}_i - w_i, \qquad \text{for} \quad 1 \leqslant i \leqslant I - 1. \tag{5.3}$$

The components Qw_i of the vector $Q\mathbf{w}$ are defined by

$$\begin{aligned} Qw_i &= \gamma_i \delta^2 w_i, \qquad \text{for} \quad 1 \leqslant i \leqslant I - 1, \\ Qw_0 &= Qw_I = 0, \end{aligned} \tag{5.4}$$

where the coefficients γ_i depend on $\mathbf{w}$ and satisfy $0 \leqslant \gamma_i < 1$. In order that the discrete analog of the conservation relation (3.6) remains satisfied after the correction (5.1), it is necessary that

$$\sum_{i=0}^{I-1} (x_{i+1} - x_i)(Qw_i + Qw_{i+1}) = 0,$$

which yields

$$\sum_{i=1}^{I-1} \gamma_i (x_{i+1} - x_{i-1})\, \delta^2 w_i = 0. \tag{5.5}$$

This condition leads to the following choice for the coefficients γ_i. Let $\mathscr{I}$ be the set of indices i which satisfy at least one of the two conditions:

$$\delta^2 w_i \cdot \delta^2 w_{i+1} < 0, \qquad \text{or} \qquad \delta^2 w_i \cdot \delta^2 w_{i-1} < 0.$$

Let $\mathscr{I}_+$ be the subset of $\mathscr{I}$ such that $\delta^2 w_i > 0$ and $\mathscr{I}_-$ be the subset of $\mathscr{I}$ such that $\delta^2 w_i < 0$. Let ω be a constant, $0 < \omega < 1$. We take

$$\gamma_i = \begin{cases} 0, & \text{if} \quad i \notin \mathscr{I}, \\ \gamma, & \text{if} \quad i \in \mathscr{I}_+, \\ \gamma', & \text{if} \quad i \in \mathscr{I}_-, \end{cases} \tag{5.6}$$

where γ and γ' are positive constants which are uniquely determined by (5.5) and the condition

$$\text{Max}\{\gamma, \gamma'\} = \omega. \tag{5.7}$$

Let

$$S = \sum_{i \in \mathscr{I}_+} (x_{i+1} - x_{i-1})\, \delta^2 w_i\,,$$

$$S' = -\sum_{i \in \mathscr{I}_-} (x_{i+1} - x_{i-1})\, \delta^2 w_i\,.$$

Then, condition (5.5) gives $\gamma S - \gamma' S' = 0$, from which we deduce

(i) $\gamma = \omega$ and $\gamma' = \omega S/S'$ if $S \leqslant S'$.

(ii) $\gamma = \omega S'/S$ and $\gamma' = \omega$ if $S \geqslant S'$.

Remark 5.1. In the case of equally spaced mesh-points and of a smooth function w, we have

$$\delta^2 w_i = \frac{1}{2}(w_{i-1} - 2w_i + w_{i+1}) \sim \frac{h^2}{2} w''(x_i),$$

where $h = x_{i+1} - x_i = x_i - x_{i-1}$. Hence, $Qw_i \sim \frac{1}{2}\gamma_i h^2 w''(x_i)$, which shows that our correction corresponds to the analog of a viscosity which is introduced separately at each time step like in a fractional step method (such a method was also used in [4]). The coefficients γ_i are chosen in such a way that this viscosity is introduced only near the points where the sign of the second derivative w'' changes, which has the effect of damping the oscillations.

Iterated Corrections

Instead of the correction (5.1) that we can write as $\tilde{\mathbf{v}}^{n+1} = (I + Q)\, \mathbf{v}^{n+1}$, where I is the identity operator, we can use the correction given by

$$\tilde{\mathbf{v}}^{n+1} = (I + Q)^{\nu}\, \mathbf{v}^{n+1}, \tag{5.8}$$

where ν is an integer $\geqslant 1$. Formula (5.8) means that we apply successively ν times the correction procedure described above.

Case of a System of Conservation Laws

We apply the foregoing procedure to each component of the vector V.

VI. Numerical Experiments

VI.1. *Burgers Equation*

We consider first the initial value problem for the single equation:

$$(\partial v/\partial t) + \tfrac{1}{2}(\partial/\partial x)(v^2) = 0, \tag{6.1}$$

with various types of initial conditions. We will use a rectangular grid and we will exhibit the effect of the pseudoviscosity described in Section VI.

(a) *Propagation of a discontinuity.* We take the initial condition

$$v^0(x) = \begin{cases} 1, & \text{for } x < 3, \\ 0, & \text{for } x > 3. \end{cases} \tag{6.2}$$

The jump condition (2.6) shows that the exact solution is

$$v(x, t) = \begin{cases} 1, & \text{for } x < 3 + t/2, \\ 0, & \text{for } x > 3 + t/2. \end{cases}$$

For the computations we replace the pure initial value problem by an equivalent mixed initial-boundary value problem relative to the domain

$$\mathscr{R} = \{(x, t); 0 < x < 20, t > 0\}.$$

We impose the boundary condition: $v(0, t) = 1$.

Figure 5 represents the exact and computed solutions at the time $t = 20$ The computations were made with rectangular elements and the mesh sizes: $\Delta x = \Delta t = 0.2$. We show the computed solution in three cases: no pseudoviscosity, simple pseudoviscosity, and iterated pseudoviscosity. In this last case, the discontinuity is spread out on three mesh-sizes only and there are no oscillations; Table I permits a more precise comparison of the computed values with the exact values of the solution. The computations were performed on an IBM 370-155. At each time-step we have solved the nonlinear system of algebraic equations by the iterative method (3.19) and we have used the following criterion to stop the iterations.

$$\operatorname*{Max}_{0 \leqslant i \leqslant I} |(v_i^{n+1,l+1} - v_i^{n+1,l})/v_i^{n+1,l+1}| \leqslant 10^{-6}. \tag{6.3}$$

Then, the number of iterations is approximately equal to 5. The computing time, which depends also on the iterated pseudoviscosity is of the order of 30 sec; no effort has been made to reduce this time, since we are primarily interested in studying the accuracy of the method.

(b) *Disappearance of a discontinuity.* We take the initial condition

$$v^0(x) = \begin{cases} 0, & \text{for } x < 3, \\ 1, & \text{for } x > 3. \end{cases} \tag{6.4}$$

It is well known (see for example [5]) that the solution of the initial value problem (6.1), (6.4) is not unique. But there exists a unique solution which satisfies a so-

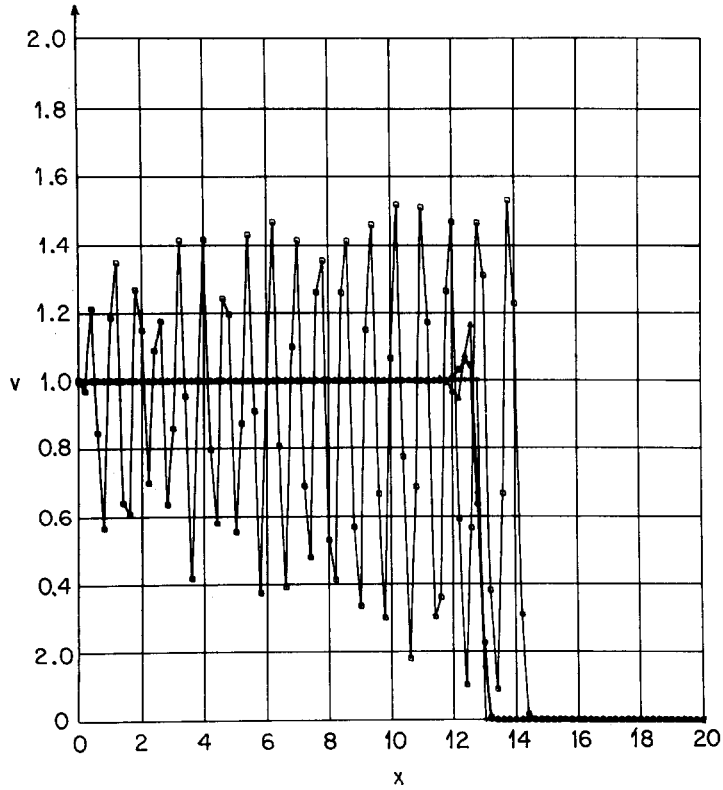

FIG. 5. Burgers equation: propagation of a discontinuity. * = exact solution; □ = computed solution with *no* pseudoviscosity; △ = computed solution with simple pseudoviscosity ($\nu = 1$, $\omega = 0.5$); ⊙ = computed solution with iterated pseudoviscosity ($\nu = 5$, $\omega = 0.7$).

TABLE I

Exact Values v and Computed Values v_h in the Neighborhood of the Discontinuity for Burgers Equation 1 (iterated pseudoviscosity: $\nu = 5$, $\omega = 0.7$)

x	v	v_h
12.0	1	1.014
12.2	1	1.020
12.4	1	1.020
12.6	1	0.983
12.8	1	0.642
13.0	0	0 307
13.2	0	0.008
13.4	0	9.6×10^{-6}
13.6	0	1.1×10^{-11}

called "entropy condition" and which is stable under small perturbations. This solution is

$$v(x, t) = \begin{cases} 0, & \text{for } x < 3, \\ (x - 3)/t, & \text{for } 3 < x < 3 + t, \\ 1, & \text{for } x > 3 + t. \end{cases} \tag{6.5}$$

This solution is the one we want to compute.

Figure 6 represents the exact and computed solutions at the time $t = 10$. The mesh-sizes are the same as previously. On the graph the curve representing the computed solution in the case of pseudoviscosity is harly discernable from the curve representing the exact solution.

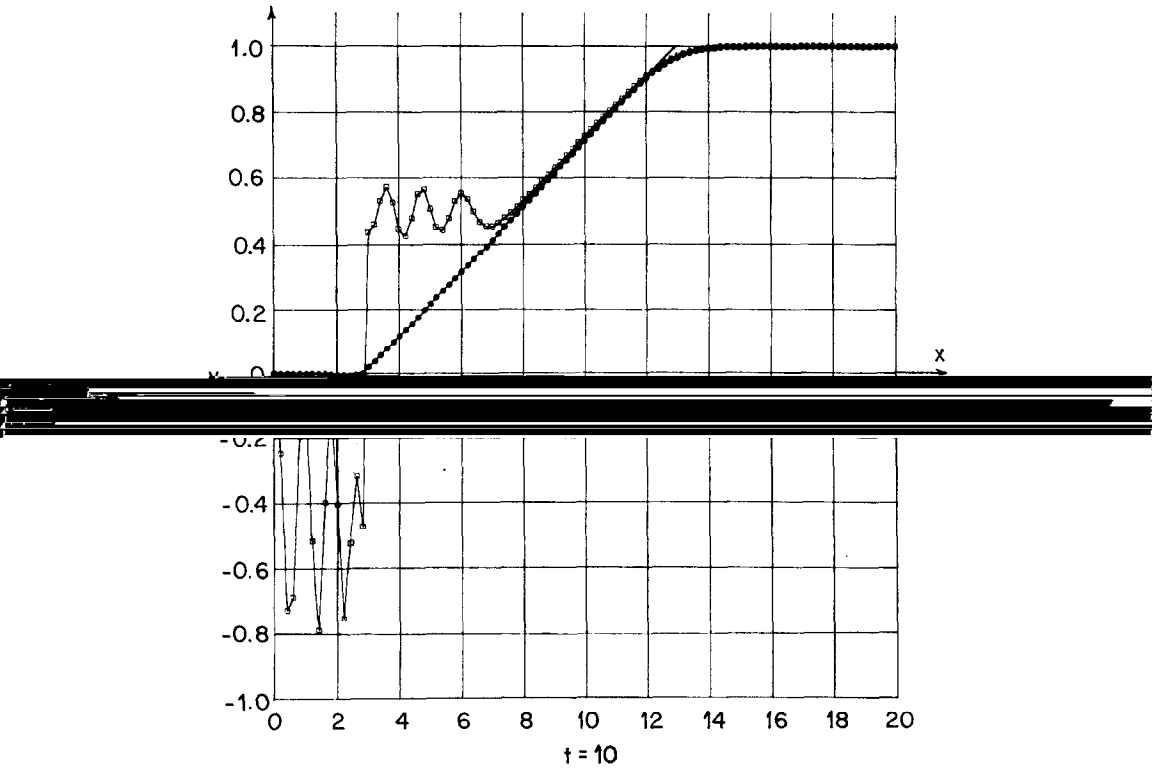

FIG. 6. Burgers equation: disappearance of a discontinuity. * = exact solution; □ = computed solution with no pseudoviscosity; ⊙ = computed solution with iterated pseudoviscosity ($\nu = 5$, $\omega = 0.5$).

VI.2. *Equations of Isentropic Flow*

We consider the isentropic flow of a polytropic gas. The equations describing the flow are (2.1) and (2.15) and the equation of state is

$$p = p^0(\rho/\rho^0)^\gamma \tag{6.6}$$

(isentropic equation of state for a polytropic gas), where we take $\gamma = 1.4, p^0 = 100$, $\rho^0 = 1.4$ (all units are C.G.S.). The initial condition is

$$\rho^0(x) = \rho^0 = 1.4, \qquad u^0(x) = 0, \qquad \forall x. \tag{6.7}$$

The boundary conditions are (2.10) and (2.11) together with (2.9); in (2.10) the function $\pi(t)$ is chosen in such a way that the corresponding function $a(t)$ which describes the movement of the free boundary is

$$a(t) = 10 + (1 - t^2)^{1/2}, \qquad \text{for} \quad 0 < t < 1. \tag{6.8}$$

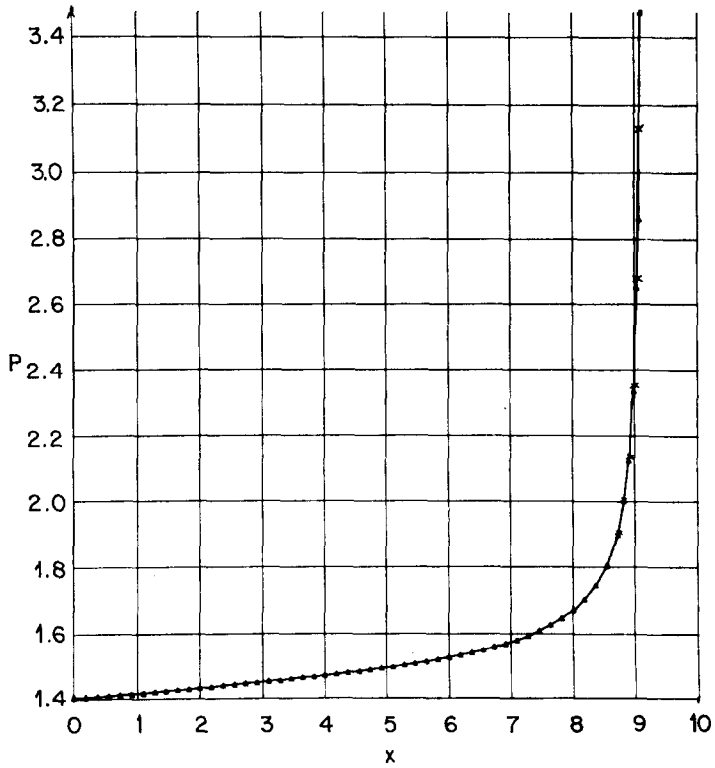

FIG. 7. Equations of isentropic flow. * = exact values of ρ; $\triangle$ = computed values of ρ.

(The function $\pi(t)$ as well as the solution V can be determined exactly by means of an analytic method based on the Rieman invariants).

To illustrate our results we show a comparison of the exact and computed values of ρ and u at the time $t = 0.995$ on Figs. 7 and 8, respectively. For the computations we have taken the initial mesh-size equal to 0.2 in the x-direction

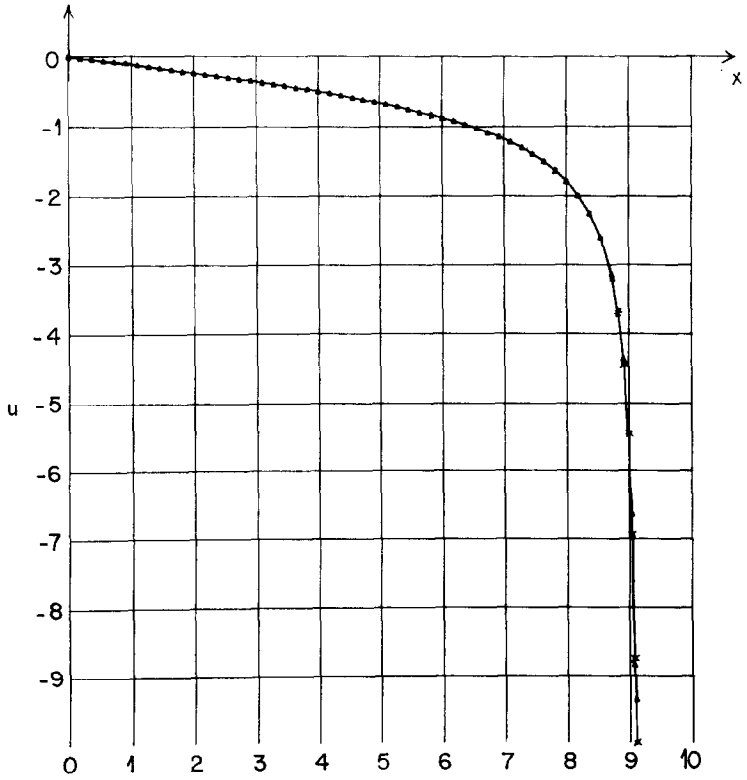

FIG. 8. Equations of isentropic flow. * = exact values of u; Δ = computed values of u.

TABLE II

Equations of Isentropic Flow

$t = 0.995$	Exact	Computed
$a'(t)$	−9.962	−9.340
$a(t) - a(0)$	−0.9002	−0.8986

Note. $a'(t)$ = speed of the free boundary; $a(t)$ = position of the free boundary.

except near the free boundary where we have subdivided the last two space-intervals in order to avoid a too large variation of the solution within each interval. At each time step we use a division of the interval $(0, a(t))$ which is similar to the initial division, i.e., $x_i^n = (a^n/a^0)\, x_i^0$. The time step is equal to 0.5×10^{-3}. We have used iterated pseudoviscosity with $\nu = 5$ and $\omega = 0.5$. For the iterative solution of the system of algebraic equations, we have used the same criterion as (6.3); the number of iterations is of the order of 5. The computation time until $t = 0.995$ is 15 min (let us remark that the computation time necessarily tends to infinity as we approach $t = 1$, because of the singularity of the solution at $t = 1$ which imposes to take smaller and smaller mesh-sizes).

On the free boundary the computed and exact values of ρ are identical since these two values are obtained directly from the boundary condition (2.10) and the equation of state (2.14). The comparison of the two corresponding values of u is particularly interesting since it shows how good the approximation of the free boundary is. These values are given on Table II.

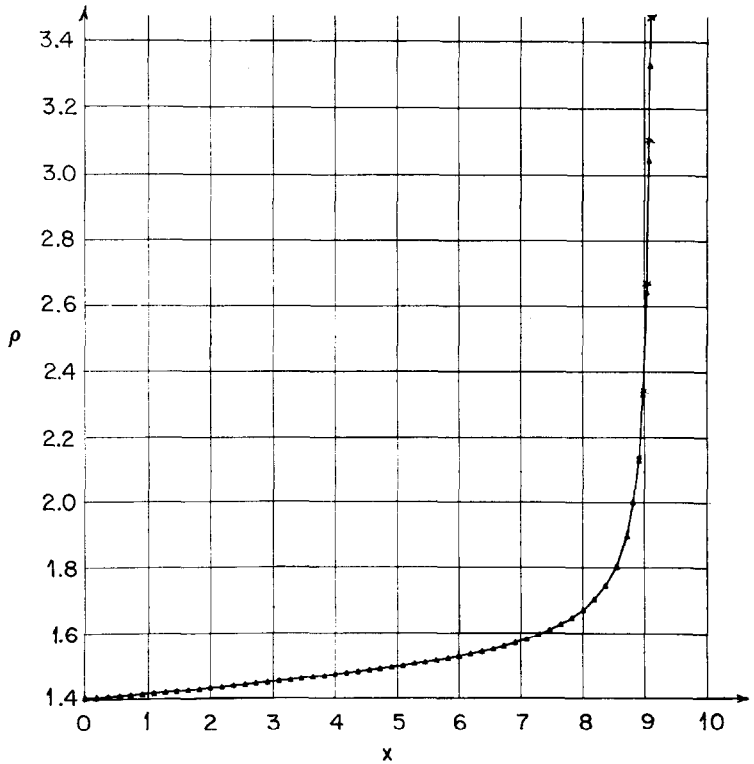

FIG. 9. Equations of general flow. * = exact values of ρ; Δ = computed values of ρ.

VI.3. *Equations of General Flow*

We consider now an arbitrary flow (isentropic or not) of the same gas as in Section VI.2. This flow is represented by the general equations (2.1) and (2.8), while the equation of state is

$$p = (\gamma - 1)\,\rho\epsilon, \tag{6.9}$$

(which is the general equation of state for a polytropic gas) with $\gamma = 1.4$.

(a) *Isentropic flow treated as a general flow.* First, we consider an isentropic flow; but, instead of using the reduced equations (2.1), (2.15), and (6.6) as we have done in Section VI.2, we use the general equations (2.1), (2.8), and (6.9), i.e., we perform the computations like in the general nonisentropic case. We take the initial condition

$$\rho^0(x) = 1.4, \qquad u^0(x) = 0, \qquad p^0(x) = 100, \tag{6.10}$$

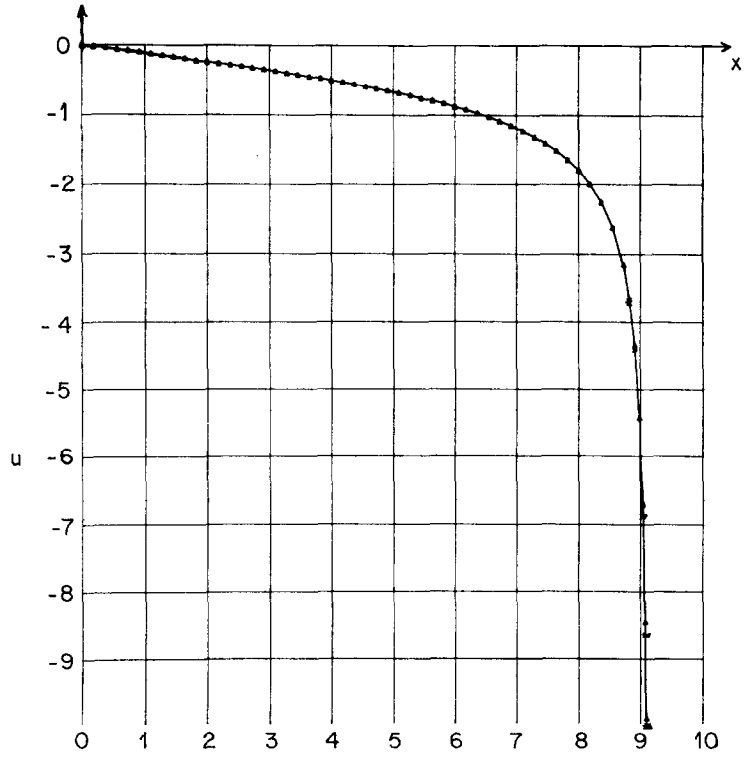

FIG. 10. Equations of general flow. * = exact values of u; Δ = computed values of u.

which yields corresponding initial values for q, ϵ, and E. Thus, the initial state of fluid is the same as in VI.2. We take also the same boundary conditions (2.10), (2.11), with the same function $\pi(t)$. It follows that the flow is identical to the isentropic flow considered in VI.2 as long as no shock appears.

We illustrate the results in the same way as in VI.2. Figures 9 and 10 show the exact and computed values of ρ and u, respectively, at the time $t = 0.995$, and Table III gives the exact and computed values of the speed $a'(t)$ and of the displacement $a(t) - a(0)$ of the free boundary. The computations were made with the same mesh-sizes as in VI.2.

TABLE III

Equations of General Flow

$t = 0.995$	Exact	Computed
$a'(t)$	−9.962	−9.873
$a(t) - a(0)$	−0.9002	−0.9001

(b) *Propagation of a shock.* We take again the initial condition (6.10) and the boundary conditions (2.10) and (2.11); but this time we choose $\pi(t) = 1000$. The exact solution corresponds to a shock which originates on the free boundary at the time $t = 0$ and propagates through the fluid at a constant speed. The state of the gas on each side of the shock is constant. The results of our computation for $t = 0.5$ are illustrated on Fig. 11 and Table IV; the mesh-sizes and the pseudo-viscosity are the same as in VI.2.

We must say that the computation of solutions with rapid variations such as those considered here and in VI.2 is possible only if we use a fine enough grid. If the grid is too coarse, we are led to computational impossibilities resulting from negative values of ρ near the free boundary.

TABLE IV

Propagation of a Shock

$t = 0.5$	Exact	Computed
$a'(t)$	−21.777	−21.769
$a(t) - a(0)$	−10.888	−10.905

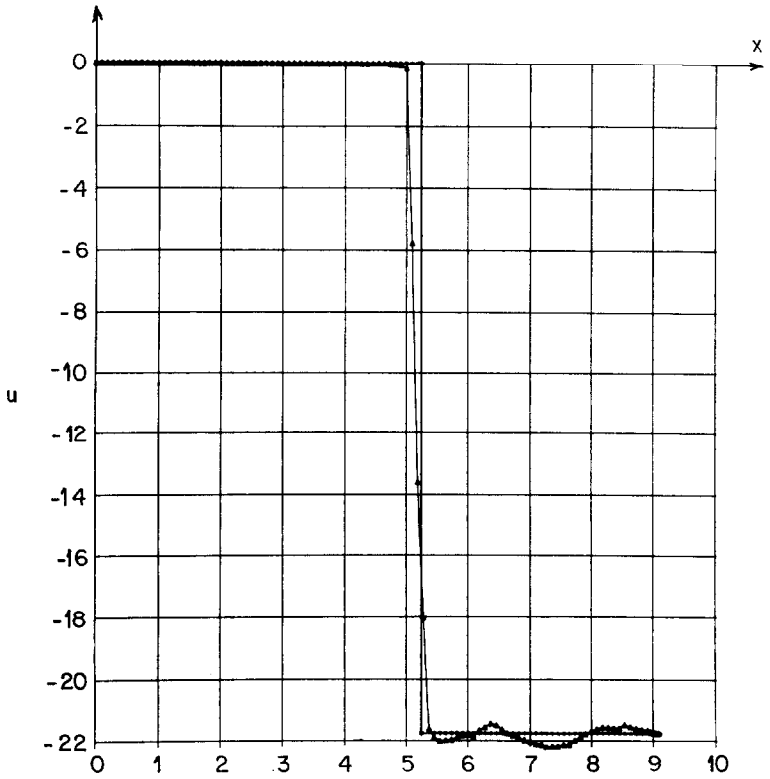

FIG. 11. Equations of general flow: shock propagation. * = exact values of u; Δ = computed values of u.

CONCLUSION

The method of space–time finite elements is primarily intended to solve multi-dimensional problems. This method must permit one to follow the free boundary and the interfaces like the Lagrangean methods and avoid the usual mesh-scrambling of these methods. However, before undertaking numerical experiments for 2-dimensional problems, we plan to experiment with several variants of the present method in the 1-dimensional case, in order to improve the computation of non-smooth solutions; in particular, we plan to experiment with discontinuous finite elements such as those recently developed by Reed and Hill [8], Lesaint and Gérin–Roze [3], Lesaint and Raviart [7] and Lesaint [6], for the transport equation. Let us also mention that we are presently performing successful numerical experiments with space–time finite elements in the 2-dimensional case, for the Stefan problem.

REFERENCES

1. A. A. AMSDEN AND C. W. HIRT, An arbitrary Lagrangian–Eulerian computer program for fluid flow at all speeds, Report LA-5100, Los Alamos Scientific Laboratory, 1973.
2. R. BONNEROT AND P. JAMET, *Int. J. Num. Met. Eng.* **8** (1974), 811–820.
3. J. GÉRIN-ROZE AND P. LESAINT, *Int. J. Num. Met. Eng.*, to appear.
4. J. L. GRAVELEAU AND P. JAMET, *SIAM J. Appl. Math.* **20** (1971), 199–223.
5. P. D. LAX, *Comm. Pure Appl. Math.* **10** (1957), 537–566.
6. P. LESAINT, These, Université de Paris VI, to appear.
7. P. LESAINT AND P. A. RAVIART, On a finite element method for solving the neutron transport equation, "Proceedings of the Symposium on Mathematical Aspects of Finite Elements in Partial Differential Equations," Madison, Wisconsin, 1974 (Carl de Boor, Ed.), Academic Press, New York.
8. W. H. REED AND T. R. HILL, Triangular mesh methods for the neutron transport equation, Report LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.
9. R. D. RICHTMYER AND K. W. MORTON, "Difference Methods for Initial-Value Problems,"